# Lecture 23

# Scalar and Vector Potentials

## 23.1 Scalar and Vector Potentials for Time-Harmonic Fields

Now that we have studied the guidance of waves by waveguides, and the trapping of electromagnetic waves by cavity resonator, it will be interesting to consider how electromagnetic waves radiate from sources. This is best done via the scalar and vector potential formulation.

Previously, we have studied the use of scalar potential $\Phi$ for electrostatic problems. Then we learnt the use of vector potential $\mathbf{A}$ for magnetostatic problems. Now, we will study the combined use of scalar and vector potentials for solving time-harmonic (electrodynamic) problems.

This is important for bridging the gap between the static regime where the frequency is zero or low, and the dynamic regime where the frequency is not low. For the dynamic regime, it is important to understand the radiation of electromagnetic fields which has a plethora of applications. Electrodynamic regime is important for studying antennas, communications, sensing, wireless power transfer applications, and many more. Hence, it is imperative that we understand how time-varying electromagnetic fields radiate from sources.

It is also crucial to understand when static or circuit (quasi-static) regimes are important. The circuit regime solves problems that have fueled the microchip and integrated circuit design (ICD) industry, and it is hence beneficial to understand when electromagnetic problems can be approximated with simple circuit problems and solved using simple laws such as Kirchhoff current law (KCL) an Kirchhoff voltage law (KVL).

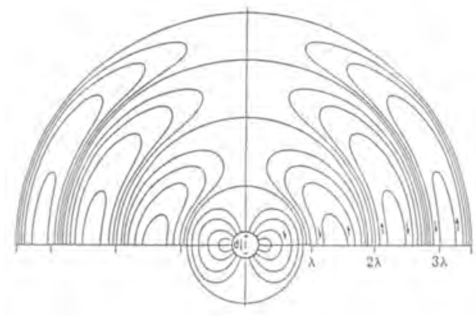


Figure 23.1: Plot of the dynamic electric field around a dipole source which is time-varying, where only the field in the upper half-space is shown. Close to the source, the field ressembles that of a static electric dipole, but far away from the source, the electromagnetic field is detached from the source: due to the finite velocity of light, the electromagnetic field cannot keep up with the changing dipole field near the source. In other words, the source starts to shed energy to the far region.

## 23.2 Scalar and Vector Potentials for Statics—A Review

Previously, we have studied scalar and vector potentials for electrostatics and magnetostatics where the frequency $\omega$ is identically zero. The four Maxwell's equations for a homogeneous medium are then

$$\nabla \times \mathbf{E} = 0 \tag{23.2.1}$$
$$\nabla \times \mathbf{H} = \mathbf{J} \tag{23.2.2}$$
$$\nabla \cdot \varepsilon \mathbf{E} = \varrho \tag{23.2.3}$$
$$\nabla \cdot \mu \mathbf{H} = 0 \tag{23.2.4}$$

Looking at the first equation above, and using the knowledge that $\nabla \times (\nabla \Phi) = 0$, we can construct a solution to (23.2.1) easily. Thus, in order to satisfy the first of Maxwell's equations or Faraday's law above, we let

$$\mathbf{E} = -\nabla \Phi \tag{23.2.5}$$

Using the above in (23.2.3), we get,

$$\nabla \cdot \varepsilon \nabla \Phi = -\varrho \tag{23.2.6}$$

Then for a homogeneous medium where $\varepsilon$ is a constant, $\nabla \cdot \varepsilon \nabla \Phi = \varepsilon \nabla \cdot \nabla \Phi = \varepsilon \nabla^2 \Phi$, and we have

$$\nabla^2 \Phi = -\frac{\varrho}{\varepsilon} \tag{23.2.7}$$

which is the Poisson's equation for electrostatics.

Now looking at (23.2.4) where $\nabla \cdot \mu \mathbf{H} = 0$, we let

$$\mu \mathbf{H} = \nabla \times \mathbf{A} \tag{23.2.8}$$

Since $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, the last of Maxwell's equations (23.2.4) is automatically satisfied. Next, using the above in the second of Maxwell's equations above, we get

$$\nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J} \tag{23.2.9}$$

Now, using the fact that $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, and Coulomb gauge that $\nabla \cdot \mathbf{A} = 0$, we arrive at

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \tag{23.2.10}$$

which is the vector Poisson's equation. Next, we will repeat the above derivation when $\omega \neq 0$.

### 23.2.1 Scalar and Vector Potentials for Electrodynamics

Since dynamic or time-varying problems are of utmost importance in electromagnetics, we will study it next. To this end, assuming linearity, we will start with frequency domain Maxwell's equations with sources $\mathbf{J}$ and $\varrho$ included, and later see how these sources $\mathbf{J}$ and $\varrho$ can radiate electromagnetic fields. Maxwell's equations in the frequency domain are

$$\nabla \times \mathbf{E} = -j\omega\mu \mathbf{H} \tag{23.2.11}$$
$$\nabla \times \mathbf{H} = j\omega\varepsilon \mathbf{E} + \mathbf{J} \tag{23.2.12}$$
$$\nabla \cdot \varepsilon \mathbf{E} = \varrho \tag{23.2.13}$$
$$\nabla \cdot \mu \mathbf{H} = 0 \tag{23.2.14}$$

In order to satisfy the last Maxwell's equation, as before, we let

$$\mu \mathbf{H} = \nabla \times \mathbf{A} \tag{23.2.15}$$

Now, using (23.2.15) in (23.2.11), we have

$$\nabla \times (\mathbf{E} + j\omega \mathbf{A}) = 0 \tag{23.2.16}$$

Since $\nabla \times (\nabla \Phi) = 0$, the above implies that $\mathbf{E} + j\omega \mathbf{A} = -\nabla \Phi$, or that

$$\mathbf{E} = -j\omega \mathbf{A} - \nabla \Phi \tag{23.2.17}$$

The above implies that the electrostatic theory of letting $\mathbf{E} = -\nabla \Phi$ we have learnt previously in Section 3.3.1 is not exactly correct when $\omega \neq 0$. The second term above, in accordance to Faraday's law, is the contribution to the electric field from the time-varying magnetic field, and hence, is termed the induction term.[1]

[1]Notice that in electrical engineering, most concepts related to magnetic fields are inductive!

Furthermore, the above shows that once $\mathbf{A}$ and $\Phi$ are known, one can determine the fields $\mathbf{H}$ and $\mathbf{E}$ assuming that $\mathbf{J}$ and $\varrho$ are given. To this end, we will derive equations for $\mathbf{A}$ and $\Phi$ in terms of the sources $\mathbf{J}$ and $\varrho$. Substituting (23.2.15) and (23.2.17) into (23.2.12) gives

$$\nabla \times \nabla \times \mathbf{A} = -j\omega\mu\varepsilon(-j\omega\mathbf{A} - \nabla\Phi) + \mu\mathbf{J} \tag{23.2.18}$$

Or upon rearrangement, after using that $\nabla \times \nabla \times \mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla \cdot \nabla\mathbf{A}$, we have

$$\nabla^2\mathbf{A} + \omega^2\mu\varepsilon\mathbf{A} = -\mu\mathbf{J} + j\omega\mu\varepsilon\nabla\Phi + \nabla\nabla \cdot \mathbf{A} \tag{23.2.19}$$

Moreover, using (23.2.17) in (23.2.14), we have

$$\nabla \cdot (j\omega\mathbf{A} + \nabla\Phi) = -\frac{\varrho}{\varepsilon} \tag{23.2.20}$$

In the above, (23.2.19) and (23.2.20) represent two equations for the two unknowns $\mathbf{A}$ and $\Phi$, expressed in terms of the known quantities, the sources $\mathbf{J}$ and $\varrho$. But these equations are coupled to each other. They look complicated and are rather unwieldy to solve at this point.

Fortunately, the above can be simplified! As in the magnetostatic case, the vector potential $\mathbf{A}$ is not unique. To show this, one can always construct a new $\mathbf{A}' = \mathbf{A} + \nabla\Psi$ that produces the same magnetic field $\mu\mathbf{H}$ via (23.2.8), since $\nabla \times (\nabla\Psi) = 0$. It is quite clear that $\mu\mathbf{H} = \nabla \times \mathbf{A} = \nabla \times \mathbf{A}'$. Moreover, one can further show that $\Phi$ is also non-unique [47]. Namely, with

$$\mathbf{A}' = \mathbf{A} + \nabla\Psi \tag{23.2.21}$$

$$\Phi' = \Phi - j\omega\Psi \tag{23.2.22}$$

it can be shown that the new $\mathbf{A}'$ and $\Phi'$ produce the same $\mathbf{E}$ and $\mathbf{H}$ field. The above is known as gauge transformation, clearly showing the non-uniqueness of $\mathbf{A}$ and $\Phi$.

To make them unique, in addition to specifying what $\nabla \times \mathbf{A}$ should be in (23.2.15), we need to specify its divergence or $\nabla \cdot \mathbf{A}$ as in the electrostatic case.[2] A clever way to specify the divergence of $\mathbf{A}$ is to choose it to simplify the complicated equations above in (23.2.19). We choose a gauge so that the last two terms in the equation (23.2.19) cancel each other. In other words, we let

$$\nabla \cdot \mathbf{A} = -j\omega\mu\varepsilon\Phi \tag{23.2.23}$$

The above is judiciously chosen so that the pertinent equations (23.2.19) and (23.2.20) will be simplified and decoupled. With the use of (23.2.23) in (23.2.19) and (23.2.20), they now become

$$\nabla^2\mathbf{A} + \omega^2\mu\varepsilon\mathbf{A} = -\mu\mathbf{J} \tag{23.2.24}$$

$$\nabla^2\Phi + \omega^2\mu\varepsilon\Phi = -\frac{\varrho}{\varepsilon} \tag{23.2.25}$$

[2]This is akin to that given a vector $\mathbf{A}$, and an arbitrary vector $\mathbf{k}$, in addition to specifying what $\mathbf{k} \times \mathbf{A}$ is, it is also necessary to specify what $\mathbf{k} \cdot \mathbf{A}$ is to uniquely specify $\mathbf{A}$.

Equation (23.2.23) is known as the Lorenz gauge[3] and the above equations are Helmholtz equations with source terms. Not only are these equations simplified, they can be solved independently of each other since they are decoupled from each other.

Equations (23.3.4) and (23.3.5) can be solved using the Green's function method we have learnt previously. Equation (23.3.4) actually constitutes three scalar equations for the three $x$, $y$, $z$ components, namely that

$$\nabla^2 A_i + \omega^2 \mu\varepsilon A_i = -\mu J_i \tag{23.2.26}$$

where $i$ above can be $x$, $y$, or $z$. Therefore, (23.3.4) and (23.3.5) together constitute four scalar equations similar to each other. Hence, we need only to solve their point-source response, or the Green's function of these equations by solving

$$\nabla^2 g(\mathbf{r}, \mathbf{r}') + \beta^2 g(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \tag{23.2.27}$$

where $\beta^2 = \omega^2 \mu\varepsilon$.

Previously, we have shown that when $\beta = 0$,

$$g(\mathbf{r}, \mathbf{r}') = g(|\mathbf{r} - \mathbf{r}'|) = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

When $\beta \neq 0$, the correct solution is

$$g(\mathbf{r}, \mathbf{r}') = g(|\mathbf{r} - \mathbf{r}'|) = \frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{23.2.28}$$

which can be verified by back substitution [33] [122][p. 26].

By using the principle of linear superposition, or convolution, the solutions to (23.3.4) and (23.3.5) are then

$$\mathbf{A}(\mathbf{r}) = \mu \iiint_V d\mathbf{r}' \mathbf{J}(\mathbf{r}') g(|\mathbf{r} - \mathbf{r}'|) = \mu \iiint_V d\mathbf{r}' \mathbf{J}(\mathbf{r}') \frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{23.2.29}$$

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon} \iiint_V d\mathbf{r}' \varrho(\mathbf{r}') g(|\mathbf{r} - \mathbf{r}'|) = \frac{1}{\varepsilon} \iiint_V d\mathbf{r}' \varrho(\mathbf{r}') \frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{23.2.30}$$

In the above $d\mathbf{r}'$ is the shorthand notation for $dx\,dy\,dz$ and hence, they are still volume integrals. The above are three-dimensional convolutional integrals in space.

### 23.2.2 More on Scalar and Vector Potentials

It is to be noted that Maxwell's equations are symmetrical and this is especially so when we add a magnetic current $\mathbf{M}$ to Maxwell's equations and magnetic charge $\varrho_m$ to Gauss's law.[4] Thus the equations then become

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} - \mathbf{M} \tag{23.2.31}$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J} \tag{23.2.32}$$

$$\nabla \cdot \mu\mathbf{H} = \varrho_m \tag{23.2.33}$$

$$\nabla \cdot \varepsilon\mathbf{E} = \varrho \tag{23.2.34}$$

[3]Please note that this Lorenz is not the same as Lorentz.

[4]In fact, Maxwell himself exploited this symmetry [39].

The above can be solved in two stages, using the principle of linear superposition because the above is a linear time invariant system. Thus, the sources of the system can be turned on and off consecutively to obtain different solutions to the system. First, we can set $\mathbf{M} = 0$, $\varrho_m = 0$, and $\mathbf{J} \neq 0$, $\varrho \neq 0$, and solve for the fields as we have done before. Second, we can set $\mathbf{J} = 0$, $\varrho = 0$ and $\mathbf{M} \neq 0$, $\varrho_m \neq 0$ and solve for the fields next. Then the total general solution, by linearity, is just the linear superposition of these two solutions.

For the second case, we set $\mathbf{J} = 0$, $\varrho = 0$ and $\mathbf{M} \neq 0$, $\varrho_m \neq 0$. Then, we can define an electric vector potential $\mathbf{F}$ such that

$$\mathbf{D} = -\nabla \times \mathbf{F} \tag{23.2.35}$$

and a magnetic scalar potential $\Phi_m$ such that

$$\mathbf{H} = -\nabla\Phi_m - j\omega\mathbf{F} \tag{23.2.36}$$

By invoking duality principle (see Section 13.2), one gather that [49]

$$\mathbf{F}(\mathbf{r}) = \varepsilon \iiint d\mathbf{r}'\mathbf{M}(\mathbf{r}')g(|\mathbf{r}-\mathbf{r}'|) = \varepsilon \iiint d\mathbf{r}'\mathbf{M}(\mathbf{r}')\frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{23.2.37}$$

$$\Phi_m(\mathbf{r}) = \frac{1}{\mu} \iiint d\mathbf{r}'\varrho_m(\mathbf{r}')g(|\mathbf{r}-\mathbf{r}'|) = \frac{1}{\mu} \iiint d\mathbf{r}'\varrho_m(\mathbf{r}')\frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{23.2.38}$$

As mentioned before, even though magnetic sources do not exist, they can be engineered. In many engineering designs, one can use fictitious magnetic sources to enrich the diversity of electromagnetic technologies.

## 23.3 When is Static Electromagnetic Theory Valid?

Now we see that the dynamic theory, with its bells and whistles, is more complicated than static theory. Also, quasi-static electromagnetic theory eventually gives rise to circuit theory and telegraphy technology. Circuit theory consists of elements like resistors, capacitors, and inductors. Given that we have now seen electromagnetic theory in its full glory, we would like to ponder when we can use simple static electromagnetics to describe electromagnetic phenomena.

We have learnt in the previous section that for electrodynamics,

$$\mathbf{E} = -\nabla\Phi - j\omega\mathbf{A} \tag{23.3.1}$$

where the second term above on the right-hand side is due to induction, or the contribution to the electric field from the time-varying magnetic filed. Hence, much things we have learnt in potential theory that $\mathbf{E} = -\nabla\Phi$ is not exactly valid. But simple potential theory that $\mathbf{E} = -\nabla\Phi$ is very useful because of its simplicity. We will study when static electromagnetic theory can be used to model electromagnetic systems.

Since the third and the fourth Maxwell's equations are derivable from the first two when $\omega \neq 0$, let us first study when we can ignore the time derivative terms in the first two of

Maxwell's equations, which, in the frequency domain, are

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{23.3.2}$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J} \tag{23.3.3}$$

When the terms multiplied by $j\omega$ above, which are associated with time derivatives, can be ignored, then electrodynamics can be replaced with static electromagnetics, which is much simpler.[5]

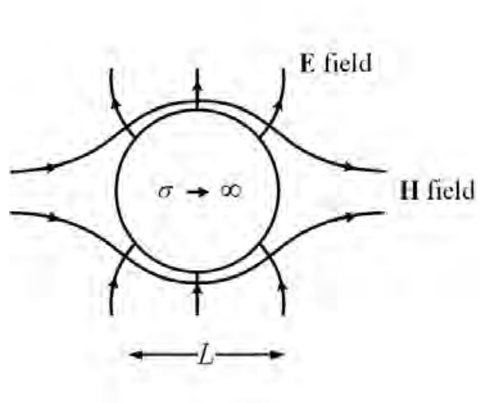


Figure 23.2: The electric and magnetic fields are great contortionists around a perfectly conducting particle. They deform themselves to satisfy the boundary conditions, $\hat{n}\times\mathbf{E} = 0$, and $\hat{n}\cdot\mathbf{H} = 0$ on the PEC surface, even when the particle is very small. In other words, the fields vary on the length-scale of $1/L$. Hence, $\nabla \sim 1/L$ which is large when $L$ is small.

### 23.3.1 Cutting Through The Chaste

To see when static electromagnetics can be used to approximate electrodynamics, we stare at the Helmholtz equations previous derived, and when they can be replaced by Poisson/Laplace equations. They are reproduced here as

$$\nabla^2\mathbf{A} + \omega^2\mu\varepsilon\mathbf{A} = -\mu\mathbf{J} \tag{23.3.4}$$

$$\nabla^2\Phi + \omega^2\mu\varepsilon\Phi = -\frac{\varrho}{\varepsilon} \tag{23.3.5}$$

By looking at Figure 23.2, in order to satisfy the boundary conditions on the wall of an object, the electromagnetic fields have to contort themselves around the object in order to satisfy the requisite boundary conditions. In order for this to happen, $\partial/\partial_x$, $\partial/\partial_y$, and $\partial/\partial_z$ are of the

[5]That is why Ampere's law, Coulomb's law, and Gauss' law were discovered first.

order of $1/L$, or that $\nabla^2$ is of the order of $1/L^2$. Hence, comparing the Laplacian operator term, which is or order $1/L^2$, and the $\beta^2 = \omega^2\mu\varepsilon$ term, if $\beta^2 L^2 \ll 1$, then Helmholtz equation can be replaced by Laplace equations, and static electromagnetics applies. In the above, $\beta = 2\pi/\lambda$ where $\lambda$ is the wavelength. Thus, if $L/\lambda \ll 1$, static theory applies. Hence, in electromagnetics, the yardstick is the wavelength. When the object size is much smaller than the wavelength, we are in the static or quasistatic regime, whereas if the object size is about the wavelength or larger, we are in the electrodynamic regime, or wave-physics regime [146].

### 23.3.2 Dimensional Analysis Approach and Coordinate Stretching[6]

To see this lucidly, it is best to write Maxwell's equations in dimensionless units or the same units. Say if we want to solve Maxwell's equations for the fields close to an object of size $L$ as shown in Figure 23.2. This object can be a small particle like the sphere, or it could be a capacitor, or an inductor, which are small; but how small should it be before we can apply static electromagnetics?

It is clear that these $\mathbf{E}$ and $\mathbf{H}$ fields will have to satisfy boundary conditions, $\hat{n} \times \mathbf{E} = 0$, and $\hat{n} \cdot \mathbf{H} = 0$ on the PEC surface, which is *de rigueur* in the vicinity of the object as shown in Figure 23.2 even when the frequency is low or the wavelength long. The fields become great contortionists in order to satisfy the boundary conditions. Hence, we do not expect a constant field around the object but that the field will vary on the length scale of $L$. So we renormalize our length scale by this length $L$ by defining a new dimensionless coordinate system such that[7]

$$x' = \frac{x}{L}, \quad y' = \frac{y}{L}, \quad z' = \frac{z}{L} \tag{23.3.6}$$

In other words, by so doing, then $Ldx' = dx$, $Ldy' = dy$, and $Ldz' = dz$, and

$$\frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial y} = \frac{1}{L}\frac{\partial}{\partial y'}, \quad \frac{\partial}{\partial z} = \frac{1}{L}\frac{\partial}{\partial z'} \tag{23.3.7}$$

Take for example a function $f(x) = \sin(\pi x/L)$ which is a periodic function varying on the length scale of $L$ having a period of $L$. Then it is quite clear that $df(x)/dx = (\pi/L)\cos(\pi x/L)$. When $L$ is small representing a rapidly varying function, $df(x)/dx \sim O(1/L)$.[8] But in the new variable, $f(Lx') = \sin(\pi x')$ and $df/dx' = \cos(\pi x')$ which is $O(1)$. In other words, in the stretched coordinate system, the field is slowly varying.

In this manner, $\nabla = \frac{1}{L}\nabla'$ where $\nabla'$ is dimensionless; or $\nabla$ will be very large when it operates on fields that vary on the length scale of very small $L$, where $\nabla'$, which is an $O(1)$ operator, will not be large because it is in coordinates normalized with respect to $L$.

Then, with coordinate stretching, the first two of Maxwell's equations become

$$\frac{1}{L}\nabla' \times \mathbf{E} = -j\omega\mu_0\mathbf{H} \tag{23.3.8}$$

$$\frac{1}{L}\nabla' \times \mathbf{H} = j\omega\varepsilon_0\mathbf{E} + \mathbf{J} \tag{23.3.9}$$

[6]This can be skipped on first reading.

[7]This dimensional analysis is often used by fluid dynamicists to study fluid flow problems [147]. This is also known as coordinate stretching [148].

[8]Reads order of $1/L$ of order $1/L$.

Here, we still have apples and oranges to compare with since **E** and **H** have different units; we cannot compare quantities if they have different units. For instance, the ratio of **E** to the **H** field has a dimension of impedance. To bring them to the same unit, we define a new $\mathbf{E}'$ such that

$$\eta_0 \mathbf{E}' = \mathbf{E} \tag{23.3.10}$$

where $\eta_0 = \sqrt{\mu_0/\varepsilon_0} \cong 377$ ohms in vacuum has the unit of impedance. In this manner, the new $\mathbf{E}'$ has the same unit as the **H** field. Then, (23.3.8) and (23.3.9) become

$$\frac{\eta_0}{L}\nabla' \times \mathbf{E}' = -j\omega\mu_0\mathbf{H} \tag{23.3.11}$$

$$\frac{1}{L}\nabla' \times \mathbf{H} = j\omega\varepsilon_0\eta_0\mathbf{E}' + \mathbf{J} \tag{23.3.12}$$

With this change, the above can be rearranged to become

$$\nabla' \times \mathbf{E}' = -j\omega\mu_0\frac{L}{\eta_0}\mathbf{H} \tag{23.3.13}$$

$$\nabla' \times \mathbf{H} = j\omega\varepsilon_0\eta_0 L\mathbf{E}' + L\mathbf{J} \tag{23.3.14}$$

By letting $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$, the above can be further simplified to become

$$\nabla' \times \mathbf{E}' = -j\frac{\omega}{c_0}L\mathbf{H} \tag{23.3.15}$$

$$\nabla' \times \mathbf{H} = j\frac{\omega}{c_0}L\mathbf{E}' + L\mathbf{J} \tag{23.3.16}$$

Notice now that in the above, $\mathbf{H}$, $\mathbf{E}'$, and $L\mathbf{J}$ have the same unit, and $\nabla'$ is dimensionless and is of order one, and $\omega L/c_0$ is also dimensionless.

Therefore, in the above, one can compare terms, and ignore the frequency dependent $j\omega$ term when

$$\frac{\omega}{c_0}L \ll 1 \tag{23.3.17}$$

The above is the same as $kL \ll 1$ where $k = \omega/c_0 = 2\pi/\lambda_0$. Thus, when

$$2\pi\frac{L}{\lambda_0} \ll 1 \tag{23.3.18}$$

the $j\omega$ terms can be ignored and the first two Maxwell's equations become static equations. Consequently, the above criteria are for the validity of the static approximation when the time-derivative terms in Maxwell's equations can be ignored. When these criteria are satisfied, then Maxwell's equations can be simplified to and approximated by the following equations

$$\nabla' \times \mathbf{E}' \doteq 0 \tag{23.3.19}$$

$$\nabla' \times \mathbf{H} \doteq L\mathbf{J} \tag{23.3.20}$$

which are the static equations, Faraday's law and Ampere's law of electromagnetic theory. They can be solved together with Gauss' laws, or the third or the fourth Maxwell's equations.

In other words, one can solve, even in optics, where $\omega$ is humongous or the wavelength very short, using static analysis if the size of the object $L$ is much smaller than the optical wavelength which is about 400 nm for blue light. Nowadays, plasmonic nano-particles of about 10 nm can be made. If the particle is small enough compared to wavelength of the light, electrostatic analysis can be used to study their interaction with light. And hence, static electromagnetic theory can be used to analyze the wave-particle interaction. This was done in one of the homeworks. Figure 23.3 shows an incident light whose wavelength is much longer than the size of the particle. The incident field induces an electric dipole moment on the particle, whose external field can be written as

$$\mathbf{E}_s = (\hat{r} 2\cos\theta + \hat{\theta}\sin\theta)\left(\frac{a}{r}\right)^3 E_s \tag{23.3.21}$$

while the incident field $\mathbf{E}_0$ and the interior field $\mathbf{E}_i$ to the particle can be expressed as

$$\mathbf{E}_0 = \hat{z}E_0 = (\hat{r}\cos\theta - \hat{\theta}\sin\theta)E_0 \tag{23.3.22}$$

$$\mathbf{E}_i = \hat{z}E_i = (\hat{r}\cos\theta - \hat{\theta}\sin\theta)E_i \tag{23.3.23}$$

By matching boundary conditions, as was done in the homework, it can be shown that

$$E_s = \frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon} E_0 \tag{23.3.24}$$

$$E_i = \frac{3\varepsilon}{\varepsilon_s + 2\varepsilon} E_0 \tag{23.3.25}$$

For a plasmonic nano-particle, the particle medium behaves like a plasma (see Lecture 8), and $\varepsilon_s$ in the above can be negative, making the denominators of the above expression very close to zero. This is the hallmark of a resonance phenomenon as we have seen in the surface plasmonic polariton, the transverse resonance condition, and the LC tank circuit. This is also the case of plasmonic nanoparticle resonance. Therefore, the amplitude of the internal and scattered fields can be very large when this happens, and the nano-particles will glitter in the presence of light. Even the ancient Romans realized this!

Figure 23.4 shows a nano-particle induced in plasmonic oscillation by a light wave. Figure 23.5 shows that different color fluids can be obtained by immersing nano-particles in fluids with different background permittivity ($\varepsilon$ in (23.3.24) and (23.3.25)) causing the plasmonic particles to resonate at different frequencies. This is because the resonance frequency of the plasmonic nanoparticle is obtained by solving $\varepsilon_s(\omega) + 2\varepsilon = 0$, which depends on the background medium, $\varepsilon$.

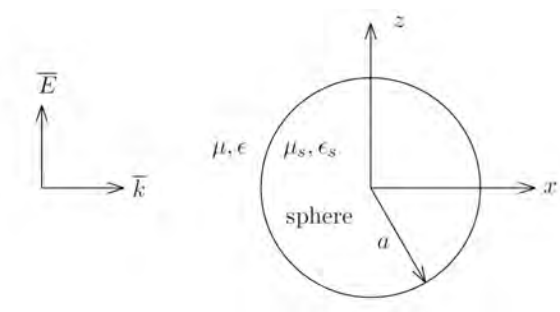


Figure 23.3: A plane electromagnetic wave incident on a particle. When the particle size is small compared to wavelength, electrostatic analysis can be used to solve this problem (courtesy of Kong [33]).

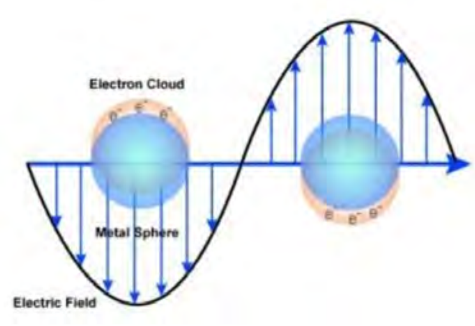


Figure 23.4: A nano-particle undergoes electromagnetic oscillation when an electromagnetic wave impinges on it. The oscillation is inordinately large when the incident wave's frequency coincides with the resonance frequency of the plasmonic particle (picture courtesy of sigmaaldric.com).

Figure 23.5: Different color fluids containing nano-particles can be obtained by changing the permittivity of the background fluids (courtesy of nanocomposix.com).

In (23.3.18), this criterion has been expressed in terms of the dimension of the object $L$ compared to the wavelength $\lambda_0$. Alternatively, we can express this criterion in terms of transit time. The transit time for an electromagnetic wave to traverse an object of size $L$ is $\tau = L/c_0$ and $\omega = 2\pi/T$ where $T$ is the period of one time-harmonic oscillation. Therefore, (23.3.17) can be re-expressed as

$$\frac{2\pi\tau}{T} \ll 1 \tag{23.3.26}$$

The above implies that if the transit time $\tau$ needed for the electromagnetic field to traverse the object of length $L$ is much small than the period of oscillation of the electromagnetic field, then static theory can be used.

The finite speed of light gives rise to delay or retardation of electromagnetic signal when it propagates through space. When this retardation effect can be ignored, then static electromagnetic theory can be used. In other words, if the speed of light had been infinite, then there would be no retardation effect, and static electromagnetic theory could always be used. Alternatively, the infinite speed of light will give rise to infinite wavelength, and criterion (23.3.18) will always be satisfied, and static theory prevails always.

### 23.3.3 Quasi-Static Electromagnetic Theory

In closing, we would like to make one more remark. The right-hand side of (23.3.13), which is Faraday's law, is essential for capturing the physical mechanism of an inductor and flux linkage. And yet, if we drop it, there will be no inductor in this world. To understand this dilemma, let us rewrite (23.3.13) in integral form, namely,

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = -j\omega\mu_0 \frac{L}{\eta_0} \iint_S d\mathbf{S} \cdot \mathbf{H} \tag{23.3.27}$$

In the inductor, the right-hand side has been amplified by multiple turns, effectively increasing $S$, the flux linkage area. Or one can think of an inductor as having a much longer effective length $L_{\text{eff}}$ when untwined so as to compensate for decreasing frequency $\omega$. Hence, the importance of flux linkage or the inductor in Faraday's law is not diminished unless $\omega = 0$.

By the same token, displacement current in (23.3.13) can be enlarged by using capacitors. In this case, even when no electric current **J** flows through the capacitor, displacement current flows and the generalized Ampere's law becomes

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = j\omega\varepsilon\eta_0 L \iint_S d\mathbf{S} \cdot \mathbf{E}' \tag{23.3.28}$$

The right-hand side can be enlarged by making $S$ large to amplify the displacement current. Thus, the displacement current in a capacitor cannot be ignored unless $\omega = 0$. Therefore, when $\omega \neq 0$, or in quasi-static case, inductors and capacitors in circuit theory are extremely important, because they amplify the flux linkage and the displacement current effects, as we shall study next. In summary, the full physics of Maxwell's equations is not lost in circuit theory: the induction term in Faraday's law, and the displacement current in Ampere's law are still retained.[9] We can still have wave phenomena in circuit theory as exemplified by the lumped element transmission-line model. In fact, by enlarging the line capacitance and line inductance, the phase velocity of the wave on such a line can be reduced making it into a slow-wave structure. That explains the success of circuit theory in electromagnetic engineering!

[9]Putatively, Maxwell got his epiphany to add displacement current to Ampere's law when he studied current through a capacitor.